# Continuity

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**Theorem 0.1** (continuity)**.** *Let $A \subset \mathbb{R}$, a function $f : A \to \mathbb{R}$ is continuous if and only if* $\forall$ open set $U \subset \mathbb{R}, \exists$ some open set $V \subset \mathbb{R}$, such that $f^{-1}(U) = V \cap A$.

*Proof.* Let $x_0 \in A$, $\varepsilon > 0$. Setting $U \subset \mathbb{R} = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $f^{-1}(U) = V \cap A$, where $V \subset \mathbb{R}$. Then $x_0 \in V$. Since $V$ is an open set, there exists $(x_0 - \delta, x_0 + \delta) \subset V$. Because $x_0 \in V \cap A$, $V \cap A$ is non-empty. Hence, $\forall x \in \mathbb{R}$ which makes $|x - x_0| < \delta, |f(x) - f(x_0)| < \varepsilon$. Thus, $\forall \varepsilon > 0, \exists \delta > 0, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.
The converse is similar and is left to the reader. ■