Yet Another Definition of Euler's Number

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1. The Start

This note begin with the question: can we use the properties of ln to obtain the tangent-line inequality algebraically? (Note that the tangent-line inequality means the inequality $\ln(x) \leq x - 1$.) Here is what I tried.

First, let $I_n = (k2^n, k2^{n+1}]$, where $n \in N$, $k \in [1, 2)$. Hence we know that this can cover $(1, +\infty)$. In fact, you can check that the same can be made for $(0, 1)$, which is left to the reader. We next show that if the inequality holds on I_n, then it holds on I_{n+1}, which is basically an induction over intervals rather than integers. Here, our inductive hypothesis is $\ln(x) \leq x - 1, \forall x \in I_n$. Hence, we have $\ln(2x) = \ln(2) + \ln(x) \leq \ln(2) + x - 1, \forall x \in I_n$. Since I_n covers $(1, +\infty)$, $x > 1 > \ln(2)$. Thus, $\ln(2x) < x + x - 1 = 2x - 1$. Since $x \in I_n$, we obtain $2x \in I_{n+1}$. Hence, $\ln(x) < x - 1, \forall x \in I_{n+1}$.

The problem is the proof of the inequality on I_0, which is $(1, 4)$.

But you will soon find it as difficult as the orignal case, which I will explain later.

2. The Climax

I thought at first that the following theorem and proof would help.

Theorem 2.1.

$$\ln(k) < k, \forall k > 0$$

Proof. We first prove that $\ln(k) < k, \forall k > 0$. Since $e > 2$, you can verify that $\ln(k) < \log_2(k)$. Thus, it suffices to show that $\log_2(k) < k$, which can be written as $k < 2^k$. We show it by induction.
1. $k \in (0, 1]$ Hence, $\log_2(k) \leq 0 < k$.
2. $\log_2(k) < k$,also $k < 2^k \wedge k > 1$ By $k > 1$ we have $k + 1 < 2k < 2 \cdot 2^k = 2^{k+1}$.

Hence, $\ln k < k, \forall k > 0$. □

The problem is very obvious. Here the inequality still holds for $y = \log_2(x)$. But in our case, the inequality is false for any other logarithm. The main problem is that this inequality depends on the property of e, so that we cannot prove it unless we use the definition of e, which makes the proof non-algebraic. Hence, our goal is impossible. Yet there is another question. Since this inequality is deeply dependent on the definition of e, is it possible to define e with this inequality? The answer is yes.

3. The End

This theorem and the following proof gives you exactly how to define e with our inequality.

Theorem 3.1. *There exists a unique smooth function $f : R_+ \to R$ with the following property:*
1. $f(xy) = f(x) + f(y)$ $(x, y > 0)$.
2. $f(x) \leq x - 1$ *; equality holds if and only if* $x = 1$.

We define e to be a positive real number such that $f(e) = 1$, and this definition is equivalent to the limit version.

Proof. We first show that f is unique. Since $f(1) = f(1 \cdot 1) = 2f(1)$, we obtain $f(1) = 0$. Apply ∂y to both sides of $f(xy) = f(x) + f(y)$, we have $xf'(xy) = f'(y)$. Let $y = 1$, we obtain $xf'(x) = f'(1)$. Hence, $f'(x) = \frac{C}{x}$, where $C = f'(1)$. Since $f(x) \leq x - 1$, we let $g(x) \coloneqq f(x) - (x - 1)$. Because the equality holds if and only if $x = 1$, we know that g has the maximum value at 1, Hence $g'(1) = 0$. It follows that $g'(x) = f'(x) - 1$. Substitute this into the previous expression to obtain $f'(1) = 1$. Hence, $f'(x) = \frac{1}{x}$. Thus, we obtain

$$f(x) = \int_1^x \frac{1}{x}\,\mathrm{d}x.$$

Hence, f is unique.

We next show that our definition of e is equivalent to the limit version. Let $F(x) \coloneqq f(xy) - f(x) - f(y)$, where y is a constant. Since $F'(x) = \frac{1}{x} - \frac{1}{x} = 0$, $F(x)$ is a constant. Because $F(1) = f(y) - f(1) - f(y) = 0$, we have $F(x) = 0$. Thus, we obtain

$$f(xy) = f(x) + f(y).$$

Since f is defined upon R_+, $f'(x) = \frac{1}{x} > 0$. Hence f is injective. Let exp be the inverse function of f. By the inverse function theorem, $\exp'(x) = \exp(x)$. By $f(xy) = f(x) + f(y)$, we have $\exp(x + y) = \exp(x)\exp(y)$. Since f is smooth, we know that exp is smooth. By continuity, we can denote exp as e^x, where $e = \exp(1)$(equivalent to $f(e) = 1$). From $f(x) \leq x - 1$, we know that $e^x \geq x + 1$, which implies $e^{\frac{1}{n}} \geq \frac{1}{n} + 1$, which is

$$e \geq \left(1 + \frac{1}{n}\right)^n.$$

Also, $e^{-\frac{1}{n+1}} \geq 1 - \frac{1}{n+1}$, which is

$$e^{-\frac{1}{n+1}} \geq \frac{n}{n+1}.$$

Thus, $e^{\frac{1}{n+1}} \leq 1 + \frac{1}{n}$. Hence, we obtain

$$\left(1 + \frac{1}{n}\right)^n \leq e \leq \left(1 + \frac{1}{n}\right)^{n+1}, \forall n \in N$$

Since f and exp are well-defined, e exists uniquely. Hence, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}$ converges. Because $\frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$ implies $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}$. By the squeeze theorem, we obtain

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

□