Infinite Product of Sums of Squares

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1. The Start

The story begins with the expression $x^2 - x^{-2}$, where $x > 0$. By the difference of squares formula, we have

$$x^2 - x^{-2} = (x + x^{-1})(x - x^{-1}) = (x + x^{-1})\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) = ...$$

Hence, let $a_i = x^{2^{-i}} + x^{-2^{-i}}$, then $a_i^2 = x^{2^{-(i-1)}} + x^{-2^{-(i-1)}} + 2 = a_{i-1} + 2$, which is $a_i = \sqrt{a_{i-1} + 2}$. Let $f(n) \coloneqq a_0 \cdot a_1 \cdot a_2 \cdot ... \cdot a_n$, $g(n) \coloneqq x^{2^{-n}} - x^{-2^{-n}}$, then $f(n) \cdot g(n) = x^2 - x^{-2}$, which immediately gives that f converges. Now the problem is to find how fast f increases and g decreases. It is obvious that we would require some approximations here.

2. The Climax

First let $t = x^2 - x^{-2}$. Thus, $f(n) \cdot g(n) = t$. Since we are making approximations for f and g, it is a good idea to apply ∂_n to both sides. Hence, we obtain $f'(n)g(n) + f(n)g'(n) = 0$, which is $f'(n) = -\frac{\mathrm{tg}'(n)}{g(n)^2}$.

Without loss of generality, set $t = 1$ and $x = e$. Thus, $g(n) = e^{2^{-n}} - e^{-2^{-n}}$, $f'(n) = -\frac{g'(n)}{g(n)^2}$. Let $m = 2^n$, $\hat{g}(m) = e^{\frac{1}{m}} - e^{-\frac{1}{m}}$. Then $\hat{f}(m) = -\frac{\hat{g}'(m)}{\hat{g}(m)^2} = \frac{\left(\frac{1}{m^2}\right)e^{\frac{1}{m}} + \frac{1}{m^2}e^{-\frac{1}{m}}}{\left(e^{\frac{1}{m}} - e^{-\frac{1}{m}}\right)^2}$. Now, it is clear how to best estimate how fast f increases and g decreases.

3. The End

The final result is here.

Theorem 3.1. *$f(n)$ increases and $g(n)$ decreases like exponentials.*

Proof. From the previous resut, we have $\hat{f}(m) = \frac{\left(\frac{1}{m^2}\right)e^{\frac{1}{m}} + \frac{1}{m^2}e^{-\frac{1}{m}}}{\left(e^{\frac{1}{m}} - e^{-\frac{1}{m}}\right)^2}$. When m is large enough, $e^{\frac{1}{m}} \approx \frac{1}{m} + 1$, $e^{-\frac{1}{m}} \approx -\frac{1}{m} + 1$. $\therefore \hat{f}'(m) \approx \frac{\frac{2}{m^2}}{\left(\frac{2}{m}\right)^2} = \frac{1}{2}$. and $\hat{g}'(m) = -\frac{1}{m^2}e^{\frac{1}{m}} - \frac{1}{m^2}e^{-\frac{1}{m}} \approx -\frac{2}{m^2}$. $\therefore \hat{f}(m) = \frac{1}{2}m$ and $\hat{g}(m) = \frac{2}{m}$. Hence, $f(n) = 2^{n-1}$ and $g(n) = 2^{1-n}$. $\square$