Finite-Dimensional Translation Spaces and Lie Group Projections

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1. The Start

This note starts with the following problem.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous non-constant function, which satisfies

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2$$

. Find all such functions.

I have the cover linear theorem in the Lie section for it. But the conclusion of that theorem depends on the choice of G, π and Φ, which weakens its power.

2. The Climax

Thus, I wanted to find another approach. And I got the idea from the standard solution: Let $g_a(x) := \frac{f(x+a)-f(x-a)}{2f(a)}$. Now, if we plug $f = \sin$ into this expression, after some calculations, we can get:

$$g = \cos$$

. The same can be done with $f = \sinh$.

In the case of $f = \sin$, $g = \cos$, it gives us a circle in a 2-dimensional space. So, we can treat this equation as a projection of some 2-dimensional curve on $\mathbb{R}$. More generally, most equations can be regarded as a projection of a 1-dimensional manifold in a high dimensional space on $\mathbb{R}$, and in which the curve can be one-parameterized by $\mathbb{R}$.

3. The End

Finally, after chosing conditions which are strong enough to make conclusions, we have the following theorem.

Theorem 3.1 (Finite-Dimensional Translation Space). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and define its translation space $V_f = \operatorname{span}_{\mathbb{R}}\{\tau_a f : a \in \mathbb{R}\}$, where $(\tau_a f)(x) = f(x+a)$.*

If $\dim V_f < \infty$, then:

1. *f is real analytic.*
2. *There exist distinct $\lambda_1, ..., \lambda_r \in \mathbb{C}$ and nonzero polynomials $P_1, ..., P_r$ such that*

$$f(x) = \sum_{j=1}^{r} P_{j(x)} e^{\lambda_j x},$$

and moreover

$$\sum_{j=1}^{r} \left(\deg P_j + 1\right) = \dim V_f.$$

In particular, $r \leq \dim V_f$*.*

3. f *is a matrix coefficient of a one-parameter subgroup of* $\mathrm{GL}\left(V_f\right)$*.*

Proof. Let $T_a : V_f \to V_f$ be the restriction of the translation operator τ_a to V_f. Since V_f is invariant and finite-dimensional, $T_a \in \mathrm{GL}\left(V_f\right)$.

The map

$$\rho : \mathbb{R} \to \mathrm{GL}\left(V_f\right), \quad \rho(a) = T_a$$

is a group homomorphism:

$$\rho(a + b) = \rho(a)\rho(b), \quad \rho(0) = I.$$

It is continuous because for every $h \in V_f$, the map $a \mapsto T_a h$ is continuous in the finite-dimensional topology of V_f.

Hence there exists a unique $B \in \mathrm{End}\left(V_f\right)$ such that

$$\rho(x) = e^{xB} \quad \forall x \in \mathbb{R}.$$

Define $L : V_f \to \mathbb{R}$ by $L(h) = h(0)$. Then

$$f(x) = (\tau_x f)(0) = L(\tau_x f) = L\left(e^{xB} f\right).$$

Now f is cyclic for B: since $V_f = \mathrm{span}\left\{e^{xB} f : x \in \mathbb{R}\right\}$, and $x \mapsto e^{xB} f$ is real analytic, its Taylor coefficients $B^k f$ span V_f. Therefore $V_f = \mathrm{span}\left\{B^k f : k \geq 0\right\}$.

Complexifying V_f, the cyclic property implies the minimal polynomial of B equals its characteristic polynomial. Thus each distinct eigenvalue $\lambda_j \in \mathbb{C}$ corresponds to exactly one Jordan block of size s_j, and

$$\sum_{j=1}^{r} s_j = \dim V_f.$$

On the block for λ_j, the matrix exponential contributes

$$e^{\lambda_j x}, x e^{\lambda_j x}, \ldots, x^{s_j - 1} e^{\lambda_j x}.$$

Applying L gives

$$f(x) = \sum_{j=1}^{r} P_{j(x)} e^{\lambda_j x},$$

where $\deg P_j = s_j - 1$. Hence

$$\sum_{j=1}^{r} \left(\deg P_j + 1\right) = \dim V_f.$$

Since $\forall \deg P_j + 1 \geq 1$,

$$r \leq \sum_{j=1}^{r} \left(\deg P_j + 1\right) = \dim V_f$$

Finally, $\{e^{xB} : x \in \mathbb{R}\}$ is a one-parameter subgroup of $\mathrm{GL}\left(V_f\right)$, and f is its matrix coefficient. ■

Finally, as a sweet treat after our hard work, let us apply this theorem to solve the initial problem.

Solution. Let $g(x) := \frac{\tau_a f(x) - \tau_{-a} f(x)}{2f(a)}$, then $g \in V_f$. Hence,

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

$$f(x+y) - f(x-y) = 2f(y)g(x).$$

Add them up, we obtain

$$f(x+y) = f(x)g(y) + f(y)g(x)$$

, which is

$$\tau_y f = g(y)f + f(y)g$$

. Since $g \in V_f$, $\tau_y f \in \mathrm{span}\{f, g\} \subset V_f$. Hence, $\dim V_f \leq 2$.

By applying **Theorem 3.1**, we obtain

$$f(x) = Ae^{\lambda x} + Be^{\mu x} \text{ or } (A + Bx)e^{\lambda x}.$$

Substitute them into the original equation, we obtain

$$f(x) = kx \text{ or } f(x) = A\sin(\omega x) \text{ or } f(x) = A\sinh(kx)$$