IMO 2002 P5

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Problem 1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that $(f(x) + f(y))(f(u) + f(v)) = f(xu - yv) + f(xv + yu)$.

Solution. Let $x = y = u = v = 0$ to obtain $4f(0)^2 = 2f(0)$, which implies $f(0) = 0$ or $\frac{1}{2}$. We consider the problem in two cases.

1. $f(0) = 0$.
   Let $u = v = 1$ to obtain $2f(1)f(x) + 2f(1)f(y) = f(x - y) + f(x + y)$. Let $x = 0$, then $2f(1)f(y) = f(y) + f(-y)$. Hence, $(2f(1) - 1)f(y) = f(-y)$. Let $x = 1, u = 1, y = v = 0$ to obtain $f(1)^2 = f(1)$.
   1. $f(1) = 0$. Then $f(x + y) + f(x - y) = 0$. Let $x = y$ to obtain $f(2x) = 0$. Hence, $f(x) \equiv 0$.
   2. $f(1) = 1$. There are two approaches. I state this case as a lemma and provide two proofs. See them below.
2. $f(0) = \frac{1}{2}$.
   Let $u = v = 1$ to obtain $2f(1)f(x) + 2f(1)f(y) = f(x - y) + f(x + y)$. Let $x = 0$, then $f(1) + 2f(1)f(y) = f(-y) + f(y)$. When $y = 0$, it follows that $2f(1) = 1$. Hence, $f(1) = \frac{1}{2}$. Thus, $f(y) + f(-y) = \frac{1}{2} + f(y)$. Therefore $f(x) \equiv \frac{1}{2}$.

Therefore, the final answers are $f(x) \equiv 0$, $f(x) \equiv \frac{1}{2}$ and $f(x) = x^2$.

Lemma 2. *Given function $f : \mathbb{R} \to \mathbb{R}$ with*
1. $(f(x) + f(y))(f(u) + f(v)) = f(xu - yv) + f(xv + yu)$
2. $f(0) = 0$
3. $f(1) = 1$

Prove that $f(x) = x^2$.

Proof. Take $u = v = 1, x = 0$ to obtain $2f(y) = f(y) + f(-y)$, which implies $f(y) = f(-y)$. Let $y = v = 0$, then $f(x)f(u) = f(xu)$. Hence, $f(x^2) = f(x)^2 \geq 0$, which implies $x \geq 0 \Rightarrow f(x) \geq 0$. Let $u = x$, $v = -y$ to obtain $(f(x) + f(y))^2 = f(x^2 + y^2)$. Let $\varphi(x) = f(\sqrt{x}), x \in [0, +\infty)$, then by multiplication, we have $\varphi(x) = \sqrt{f(x)} \geq 0$ and $\varphi(x)\varphi(y) = \varphi(xy)$. Hence, $\varphi(x^2) + \varphi(y^2) = \varphi(x^2 + y^2)$. Since $\varphi(x) \geq 0\ \forall x \in [0, +\infty)$, we obtain φ to be monotone on its domain. Hence, $\forall x \geq 0, \varphi(x) = kx$, where $k \in \mathbb{R}$. Since $f(1) = 1$, it follows that $k = 1$. Thus, $\forall x \geq 0, \varphi(x) = x$. Therefore $\forall x \geq 0, f(x) = x^2$. By $f(y) = f(-y)$, we obtain $f(x) = x^2$. □

Proof. Let $u = v = 1$, then $2f(x) + 2f(y) = f(x + y) + f(x - y)$. Let $x = 0$, it follows that $f(y) = f(-y)$. Take $y = 1$ to obtain $f(x + 1) = 2f(x) + 2 - f(x - 1)$. Since $f(0) = 0$, $f(1) = 1$, by induction, $f(n + 1) = 2f(n) + 2 - f(n - 1) = 2n^2 + 2 - (n - 1)^2 = (n + 1)^2$ implies $f(n) = n^2\ \forall n \in \mathbb{N}$. By $f(y) = f(-y)$, it follows that $f(n) =$

n^2 $\forall n \in \mathbb{Z}$. Let $y = v = 0$ to obtain $f(x)f(u) = f(xu)$, then it is not hard to show that $f(q) = q^2$ $\forall q \in \mathbb{Q}$. Since $f(x)f(u) = f(xu)$, $f(x^2) = f(x)^2 \geq 0$, which implies $x \geq 0 \Rightarrow f(x) \geq 0$. Let $x, y \geq 0$, $u = x$, $v = -y$ to obtain $f(x^2 + y^2) = (f(x) + f(y))^2 = f(x)^2 + 2f(x)f(y) + f(y)^2 = f(x^2) + 2f(x)f(y) + f(y)^2 \geq f(x^2)$. Hence, f is monotone on $[0, +\infty)$. Hence, $\forall x \geq 0, f(x) = x^2$. By $f(x) = f(-x)$, it follows immediately that $f(x) = x^2$. $\square$