IMO 1992 P2

Table of Contents

Problem 1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 + f(y)) = y + f(x)^2$.

Note 2. We denote $f^2(x) = f(f(x))$.

Solution. Let $x = 0$, then $f^2(y) = y + f(0)^2$. Hence, f^2 is bijective, which implies f is bijective. Hence, take $t \in \mathbb{R}$ such that $f(t) = 0$. Let $a = f^2(t) = f(0)$. Let $x = t$, $y = 0$, then $f(t^2 + a) = f(t)^2 = 0 = f(t)$. By bijectivity, we obtain $t^2 + a = t$. Let $x = 0$, $y = t$, it follows that $a = f^2(t) = t + f(0)^2 = t + a^2$. Substitute $t = t^2 + a$ in to obtain $t^2 + a^2 = 0$. Since $t^2 \geq 0$ and $a^2 \geq 0$, it follows that $t = 0$. Thus, $f(0) = 0$. Let $y = 0$ to obtain $f(x^2) = f(x)^2$. Let $a = x^2, b = f(y)$, then $a \in [0, +\infty), b \in \mathbb{R}$, and $f(a + b) = y + f(x)^2 = f(x^2) + f(f(y)) = f(a) + f(b)$. Then f is odd. Hence, $f(x + y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$. Since $f(x^2) = f(x)^2 \geq 0$, $f(x)$ is monotone, then $f(x) = kx$. Substitute this in to obtain $kx^2 + k^2y = y + k^2x^2$, then $k^2 - k = 0$ and $k^2 = 1$. Thus, $k = 1$, which implies $f(x) = x$.