Functional Equations with Finitely Many Ratios

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Problem 1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
1. $f(x+y) + f(x-y) = 2f(x) + 2f(y)$
2. $f(xy) = f(x)f(y)$
3. $f(1) = 1$
4. $\forall x \in \mathbb{R}, \exists n \in \{1, 2, ..., 10\}$ such that $f(x) = nx^2$.

Solution. It's not hard to see that the final answer is $f(x) = x^2$. Next we prove it.
Let $x = 0$, $y = 0$, then $2f(0) = 4f(0)$. Hence, $f(0) = 0$. Let $x = 0$, we obtain $f(y) + f(-y) = 2f(y)$. Thus, $f(y) = f(-y)$. Therefore $f(-1) = 1$.
Then we will show that $f(n) = n^2 \ \forall n \in \mathbb{Z}$. By $f(y) = f(-y)$, it suffices to show that $f(n) = n^2 \ \forall n \in \mathbb{N}$. Since $f(0) = 0$ and $f(1) = 1^2$ are already known, by induction, we obtain $f(n) = f(n-1+1) = -f(n-2) + 2f(n-1) + 2f(1) = -(n-2)^2 + 2(n-1)^2 + 2 = n^2$.
Next, suppose $f(x) = \varphi(x)x^2$. We stipulate that $\varphi(0) = 1$, since $f(0) = 0$. Then we obtain $\varphi : \mathbb{R} \to \{1, 2, ..., 10\}$. We study the case in which $x \neq 0$, since we already know that $f(0) = 0$. By property 1, we obtain $\varphi(x+y)(x+y)^2 + \varphi(x-y)(x-y)^2 = 2\varphi(x)x^2 + 2\varphi(y)y^2$. Simplify this expression to obtain $(\varphi(x+y) + \varphi(x-y) - 2\varphi(x))x^2 + (\varphi(x+y) + \varphi(x-y) - 2\varphi(y))y^2 + 2(\varphi(x+y) - \varphi(x-y))xy = 0$. Then we will show that $\exists y \in \mathbb{Z}$ such that $\varphi(x) = \varphi(y)$. Let $a = \varphi(x+y) + \varphi(x-y) - 2\varphi(x)$, $b = \varphi(x+y) + \varphi(x-y) - 2\varphi(y)$, $c = 2\varphi(x+y) - 2\varphi(x-y)$, then $ax^2 + by^2 + cxy = 0$. Since φ is a bounded integer-valued function, we know that there are only finitely many possibilities of (a, b, c). Thus, this equation gives at most finitely many solutions of y satisfying this condition for a given x when a, b, c is not all zero. But $y \in \mathbb{Z}$ can take infinitely many values. Thus, $\exists y$ such that $a = b = c = 0$ for a given x. In this case, we obtain $\varphi(x+y) + \varphi(x-y) = 2\varphi(x) = 2\varphi(y)$, which implies $\varphi(x) = \varphi(y)$.

Last, since $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, \varphi(x) = \varphi(y)$ and $\varphi(x) = 1 \ \forall x \in \mathbb{Z}$, we obtain $\varphi(x) = 1 \ \forall x \in \mathbb{R}$. Thus, $f(x) = x^2$.