Exchanging Exponentials

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1. The Start

The problem we are discussing here can be formalized with the following proposition.

Proposition 1.1. *Given* $1 < a < b$, $a^b < b^a$ *if and only if* $f(a,b) > 0$.

Here, the map $f(a,b)$ is our purpose.

It is a natural thought to use exponentials like a^a or b^b to solve the problem. However, this is not effective. Here is an example.

Solution. $a^a < a^b < b^b$, while $b^a < b^b$.

Hence we need some other tools, instead of pure inequalities. Another natural thought to solve the problem, is to use the natural logarithm. Thus the problem is asking when $e^{b\ln(a)} < e^{a\ln(b)}$ is true, which is in fact simply $b\ln(a) < a\ln(b)$. It is a simple matter to see that this is equivalent to $\frac{a}{b} < \frac{\ln(a)}{\ln(b)}$. However, you can't see how can this solve the problem. A traditional way is to rewrite the inequality like the following:

$$\frac{\ln(a)}{a} < \frac{\ln(b)}{b}.$$

The next step is to let $f(x) := \frac{\ln(x)}{x}$, and study the property of this function on the interval $(1, +\inf)$. This, as you can see, is pretty boring. After drawing many pictures, which filled half of my crafts, I discovered the following, which is much more interesting.

2. The Climax

We have already shown that the original problem is equivalent to find when the equality $\frac{a}{b} < \frac{\ln(a)}{\ln(b)}$ holds. Let's draw a picture of $x \mapsto x$ and $x \mapsto \ln x$ on a same figure. Hence we are actually comparing the the slope of the line passing through $(a, \ln(a))$ and $(b, \ln(b))$. Since these are lines, we may let the one through the points $(a, \ln(a))$ and $(b, \ln(b))$ be $y = m * x + n$. By Karamata's inequality, we know that the original inequality holds if and only if $n < 0$.

3. The End

The following theorem is the final result of this note.

Theorem 3.1. *Given* $1 < a < b$, $a^b < b^a$ *if and only if* $a < e^{\frac{\ln(k)}{k-1}}$, *where* $k = \frac{b}{a}$ *And the inequality never holds if* $a \geq e$.

Proof. First proof that $a^b < b^a \Rightarrow a < e^{\frac{\ln(k)}{k-1}}$. Since $a^b < b^a$, we obtain $e^{b\ln(a)} < e^{a\ln(b)}$. By the results above, we have

$$\frac{a}{b} > \frac{\ln(a)}{\ln(b)}.$$

Set points $A(a, \ln(a))$ and $B(b, \ln(b))$ in a plane. Let the line through A and B be $y = mx + n$. It is clear that $A(a, ma + n)$, $B(b, mb + n)$. Thus, by Karamata's inequality, the original inequality holds if and only if $n < 0$. So we only need to prove that $n < 0$ if and only if $a < e^{\frac{\ln(k)}{k-1}}$. Substitute the position of A, B into $\mathrm{AB} : y = mx + n$, we obtain

$$m = \ln(a) - \frac{\ln\left(\frac{b}{a}\right)}{\frac{b}{a} - 1}.$$

Let $k = \frac{b}{a}$, then $m = \ln(a) - \frac{\ln(k)}{k-1}$. Thus, $a < e^{\frac{\ln(k)}{k-1}}$.

Finally, we prove that the inequality never holds if $a \geq e$. It is obvious that $\ln(k) \leq k - 1$. Clearly, since $b > a$, $k > 1$. Thus we obtain $\frac{\ln(k)}{k-1} \leq 1$. Hence, unless $a < e$, otherwise the inequality is false, which is equivalent to our goal. ∎