Taylor Form

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1. The Start

The reason why I call this expression $\sum_{i=0}^{\infty} a_i \omega^i (\mathrm{d}\omega)^k$, where ω is a smooth 0-form, is not hard to see. Then an interesting question is raised: does this form act exactly like Taylor series, which only converges for certain kinds of functions. In this case, that is to say, does this expression converge only for certain kinds of differential forms?

1.1. The Climax

It's obvious that by choosing certain a_i and k we can control the degree of this expression to be any non-negative number. In addition, it is not hard to see that this proposition must not be true, otherwise in the case of $\mathbb{R}$ this would immediately imply that $C^\omega(\mathbb{R}) = C^\infty(\mathbb{R})$, which is obviously false. The key is to calculate this expression in a smooth coordinate chart. But for that we would have to study $\sum_{i=0}^{\infty} a_i \varphi^i$ first. Hence, we have the following lemma.

Lemma 1.1.1. *There exist no $x_0 \in \mathbb{R}$ and no smooth function φ defined on a neighborhood of x_0 such that for any $f \in C^\infty(\mathbb{R})$, there exist some real numbers a_i and a neighborhood U_f of x_0 with $f(x) = \sum_{i=0}^{\infty} a_i \varphi(x)^i$ for all $x \in U_f$.*

Proof. Assume that such x_0 and φ exist. Take $f(x) = x$. By hypothesis there exist a_i and a neighborhood U_1 of x_0 with $x = \sum_{i=0}^{\infty} a_i \varphi(x)^i$ on U_1. If φ is constant on U_1, the right-hand side is constant, contradiction. If φ is not injective on U_1, take $x_1 \neq x_2$ in U_1 with $\varphi(x_1) = \varphi(x_2)$; the right-hand side takes the same value at x_1 and x_2, contradiction. Hence φ is injective on U_1, so it is strictly monotone there. Let $I = \varphi(U_1)$ and $\Phi = \varphi^{-1} : I \to U_1$. If $\varphi'(x_0) = 0$, then Φ cannot be differentiable at $y_0 = \varphi(x_0)$. But taking $f(x) = x - x_0$, we get $\Phi(y) - x_0 = \sum_{i=0}^{\infty} a_i y^i$ on a neighborhood of y_0, so Φ is real analytic there, contradiction. Hence $\varphi'(x_0) \neq 0$, so Φ is smooth near y_0. Now take a nonzero smooth function f which is flat at x_0, i.e. $f^{(j)}(x_0) = 0$ for all $j \in \mathbb{N}$. By hypothesis there exist b_i and a neighborhood U_2 of x_0 with $f = \sum_{i=0}^{\infty} b_i \varphi^i$ on U_2. On $U_1 \cap U_2$ we have $f \circ \Phi = \sum_{i=0}^{\infty} b_i y^i$, which is analytic near y_0. Since f is flat at x_0 and Φ is smooth, $f \circ \Phi$ is flat at y_0. An analytic function which is flat at a point vanishes identically near that point. Hence $f \circ \Phi \equiv 0$ near y_0, so $f \equiv 0$ near x_0, contradiction. Thus no such x_0 and φ exist. □

Now, we are ready to prove the final theorem.

1.2. The End

Throughout, a series of the form $\sum_{i=0}^{\infty} a_i \omega^i (\mathrm{d}\omega)^k$ is understood to converge locally uniformly together with all termwise derivatives, so that termwise differentiation and integration are legitimate.

Theorem 1.2.1. *Let M be a smooth manifold with $\dim M \geq 1$. It is impossible to find a smooth 0-form ω such that $\forall k \in \mathbb{N}_+$ with $k \leq \dim M, \forall f \in \Omega^k(M), \exists a_i \in \mathbb{R}$, such that $f = \sum_{i=0}^{\infty} a_i \omega^i (\mathrm{d}\omega)^k$.*

Proof. Assume that there exists some $\omega \in \Omega^0(M)$ such that forall $k \in \mathbb{N}_+$ with $k \leq \dim M$, $\forall f \in \Omega^k(M), \exists a_i \in \mathbb{R}$, $f = \sum_{i=0}^{\infty} a_i \omega^i (\mathrm{d}\omega)^k$.

It suffices to take $k = 1$. Given a smooth chart (U, φ), we consider the case of smooth 1-forms. Then f on U is $\sum_{i=1}^{n} f_i \,\mathrm{d}x^i$, $\mathrm{d}\omega$ on U is $\sum_{i=1}^{n} \frac{\partial \omega}{\partial x^i} \,\mathrm{d}x^i$. For the ith coordinate, it follows that $f = \sum_{j=0}^{\infty} a_j \omega^j \,\mathrm{d}\omega \Rightarrow f_i = \frac{\partial \omega}{\partial x^i} \sum_{j=0}^{\infty} a_j \omega^j$. We claim that $\frac{\partial \omega}{\partial x^i}$ never vanishes on U. Indeed, for any $p \in U$, take $\chi \in C_c^\infty(U)$ equal to 1 near p and set $f = \chi \,\mathrm{d}x^i$; this is a global smooth 1-form on M. Comparing the $\mathrm{d}x^i$ components on a neighborhood of p gives $1 = \frac{\partial \omega}{\partial x^i} \sum_{j=0}^{\infty} a_j \omega^j$, so $\frac{\partial \omega}{\partial x^i}(p) \neq 0$. Since p is arbitrary, $\frac{\partial \omega}{\partial x^i}$ is nowhere vanishing on U.

Fix $x' = (x^1, ..., x^{i-1}, x^{i+1}, ..., x^n)$ and t_0 with $(t_0, x') \in U$. Take a cutoff function $\chi \in C_c^\infty(U)$ equal to 1 near (t_0, x'), and write $t = x^i$. For any $h \in C^\infty(\mathbb{R})$, set $f = \chi h'(t) \,\mathrm{d}x^i$; this is a global smooth 1-form on M. On the neighborhood where $\chi = 1$, comparing the $\mathrm{d}x^i$ components gives $h'(t) = \frac{\partial \omega}{\partial x^i}(t, x') \sum_{j=0}^{\infty} a_j \omega(t, x')^j$. Since $\frac{\mathrm{d}}{\mathrm{d}t} \omega(t, x')^{j+1} = (j+1)\omega(t, x')^j \frac{\partial \omega}{\partial x^i}(t, x')$, integrating in t near t_0 gives $h(t) = C + \sum_{j=0}^{\infty} \frac{a_j}{j+1} \omega(t, x')^{j+1}$ for some constant $C \in \mathbb{R}$. Setting $b_0 = C$ and $b_{j+1} = \frac{a_j}{j+1}$ for $j \geq 0$, we get $h(t) = \sum_{j=0}^{\infty} b_j \omega(t, x')^j$ near t_0. Thus, with $\psi(t) = \omega(t, x')$ defined on a neighborhood of t_0, every $h \in C^\infty(\mathbb{R})$ is a power series in ψ near t_0, which contradicts the lemma above. □