Exterior Derivatives via Linear Algebra

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1. The Start

This note starts with the exterior derivative in Euclidean space, which is the following.

Example. Derive the 2-form of the external differential by defining it in Euclidean space.

Solution. Let $\omega = Pdx + Qdy + Rdz$, we deduce that

$$dP = \partial P_x dx + \partial P_y dy + \partial P_z dz$$

$$dQ = \partial Q_x dx + \partial Q_y dy + \partial Q_z dz$$

$$dR = \partial R_x dx + \partial R_y dy + \partial R_z dz$$

$$d\omega = dP \wedge dx + dQ \wedge dy + dR \wedge dz$$

. Expand item by item to obtain

$$dP \wedge dx = \partial P_y dy \wedge dx + \partial P_z dz \wedge dx$$

$$dQ \wedge dy = \partial Q_x dx \wedge dy + \partial Q_z dz \wedge dy$$

$$dR \wedge dz = \partial R_x dx \wedge dz + \partial R_y dy \wedge dz$$

. Substitute them into the expression of $d\omega$, we obtain that

$$d\omega = \left(\partial R_y - \partial Q_z\right) dy \wedge dz + (\partial P_z - \partial R_x) dz \wedge dx + \left(\partial Q_x - \partial P_y\right) dx \wedge dy$$

. And we are done.

2. The Climax

But this solution is extremely ugly. We obtain that $\omega = \begin{pmatrix} P & Q & R \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$. So is there a way to solve the problem through linear algebra? Let's have a try.

Solution. We obtain that for any function f, $df = \begin{pmatrix} \partial f_x & \partial f_y & \partial f_z \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$. Let $\omega = \begin{pmatrix} P & Q & R \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$, then $d\omega = \begin{pmatrix} dP & dQ & dR \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$.

Obtain that $\begin{pmatrix} dP & dQ & dR \end{pmatrix} = \begin{pmatrix} dx & dy & dz \end{pmatrix} \begin{pmatrix} \partial P_x & \partial P_y & \partial P_z \\ \partial Q_x & \partial Q_y & \partial Q_z \\ \partial R_x & \partial R_y & \partial R_z \end{pmatrix}$. Substitute this in, thus, $d\omega = \begin{pmatrix} dx & dy & dz \end{pmatrix} \begin{pmatrix} \partial P_x & \partial P_y & \partial P_z \\ \partial Q_x & \partial Q_y & \partial Q_z \\ \partial R_x & \partial R_y & \partial R_z \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$.

Let $\vec{v} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} \partial P_x & \partial P_y & \partial P_z \\ \partial Q_x & \partial Q_y & \partial Q_z \\ \partial R_x & \partial R_y & \partial R_z \end{pmatrix}$, then $d\omega = \vec{v}^T \boldsymbol{A} \vec{v}$.

Here, we may need the following lemma.

Lemma 2.1. *Let R be an exchange ring, A is combination algebra on R. Let $\vec{v} \in A^n$, for any matrix $\boldsymbol{S} = \left(s_{ij}\right) \in M_n(R)$, If $Q = \vec{v}^T \boldsymbol{S} \vec{v}$, then $Q = \sum_{i=1}^n \sum_{j=1}^n s_{ij} v_i v_j$.*

Since $\forall dx, dy, dx \wedge dx = 0, dx \wedge dy = -dy \wedge dx$, after applying **Lemma 2.1**, we obtain that:

$$d\omega = \left(\partial R_y - \partial Q_z\right) dy \wedge dz + (\partial P_z - \partial R_x) dz \wedge dx + \left(\partial Q_x - \partial P_y\right) dx \wedge dy$$

. And we are done.

3. The End

Now, the only work is to prove **Lemma 2.1**, and here it is.

Proof. What are you looking at? It's just some dumb, straightforward calculation — do it. ∎