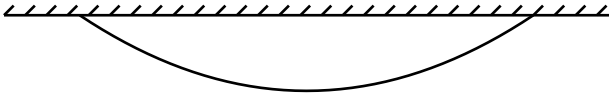
# Derivation of the Catenary Equation

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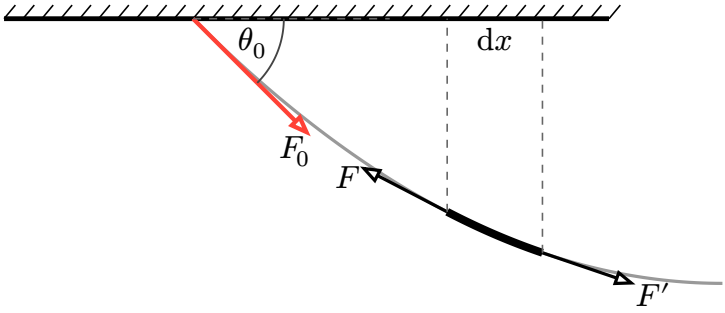
This, is a catenary.



It is not so obvious that this is not a parabola. But I will expalin here. Suppose that there is a ball being thrown out(note that we are not considering resistance here). Since there is no resistance, we know that its velocity on the horizontal direction is a constant. Therefore, if we make its path, which is a parabola, a rope, then its mass is uniform on the vertical direction. The problem is, in the case of catenary, the mass is uniform over the rope itself, not over the vertical direction. Thus, a catenary is not a parabola.

But what is it? Here is a solution.

*Solution.*




As shown in the figure, take a small segment of the heavy rope corresponding to $\mathrm{d}x$. its mass is

$$\mathrm{d}m = \lambda\,\mathrm{d}l = \lambda\frac{\mathrm{d}x}{\cos(\theta)}. \tag{1}$$

Meanwhile, we obtain the following equations.

$$F\cos\theta = F'\cos(\theta-\mathrm{d}\theta) = \frac{mg}{\tan\theta_0}, \tag{2}$$

$$F\sin\theta - F'\sin(\theta-\mathrm{d}\theta) = \mathrm{d}mg, \tag{3}$$

where

$$\cos(\theta-\mathrm{d}\theta) = \cos\theta\cos\mathrm{d}\theta + \sin\theta\sin\mathrm{d}\theta \approx \cos\theta + \mathrm{d}\theta\sin\theta, \tag{4}$$

$$\sin(\theta-\mathrm{d}\theta) = \sin\theta\cos\mathrm{d}\theta - \cos\theta\sin\mathrm{d}\theta \approx \sin\theta - \mathrm{d}\theta\cos\theta. \tag{5}$$

Substitute (4) into (2) and (5) into (3) to obtain

$$F\cos\theta = F'\cos\theta + \mathrm{d}\theta F'\sin\theta = \frac{mg}{\tan\theta_0} \tag{6}$$

$$F\sin\theta - F'\sin\theta + \mathrm{d}\theta F'\cos\theta = \mathrm{d}mg \tag{7}$$

By (6) we have $F'\cos\theta = F\cos\theta - \mathrm{d}\theta F'\sin\theta$. Multiply by $\frac{\sin\theta}{\cos\theta}$ on both sides to obtain

$$F'\sin\theta = F\sin\theta - \mathrm{d}\theta F'\frac{\sin^2\theta}{\cos\theta}. \tag{8}$$

Substitute (8) into (7), we obtain

$$F\sin\theta - F\sin\theta + \mathrm{d}\theta F'\frac{\sin^2\theta}{\cos\theta} + \mathrm{d}\theta F'\cos\theta = \mathrm{d}mg,$$

which is

$$d\theta\frac{F'}{\cos\theta} = \mathrm{d}mg. \tag{9}$$

Substitute (1) into (9) to obtain

$$\mathrm{d}\theta\frac{F'}{\cos\theta} = \frac{\lambda\,\mathrm{d}xg}{\cos\theta},$$

which is

$$\mathrm{d}\theta F' = \lambda\,\mathrm{d}xg.$$

Hence, we obtain

$$\mathrm{d}\theta F'\cos\theta = \lambda\,\mathrm{d}xg\cos\theta.$$

Since $F \approx F'$, we have

$$\mathrm{d}\theta F\cos\theta = \lambda\,\mathrm{d}xg\cos\theta. \tag{10}$$

Substitute (6) into (10) to obtain

$$\mathrm{d}\theta\frac{mg}{\tan\theta_0} = \lambda\,\mathrm{d}xg\cos\theta,$$

which is

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\lambda}{m}\tan\theta_0\cos\theta. \tag{11}$$

Let $k = \frac{\lambda}{m}\tan\theta_0$, we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = k\cos\theta. \tag{12}$$

It is obvious that

$$\tan\theta = \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{13}$$

By applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of (13), we obtain that

$$\frac{1}{\cos^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}. \tag{14}$$

Substitute (12) in, we have

$$\frac{k}{\cos\theta} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}. \tag{15}$$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta} = \sqrt{\frac{1}{\cos^2\theta} - 1}$, we have

$$\cos\theta = \frac{1}{\sqrt{\tan^2\theta + 1}}. \tag{16}$$

Substitute (16) into (15), we have

$$k\sqrt{\tan^2\theta + 1} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}. \tag{17}$$

Substitute (13) in, we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = k\sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1}. \tag{18}$$

Take $\varphi = \frac{\mathrm{d}y}{\mathrm{d}x}$, we have

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = k\sqrt{\varphi^2 + 1}. \tag{19}$$

The solution to this equation is $\varphi = \sinh(kx)$.

Hence, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh(kx). \tag{20}$$

The solution is $y = \frac{1}{k}\cosh(kx) + C$.

Let the lowest point be $(0, 0)$, we obtain

$$y = \frac{1}{k}\cosh(kx) - \frac{1}{k}.$$

Hence, the form of a catenary is

$$y = \frac{1}{k}\cosh(kx) - \frac{1}{k},$$

where $k = \frac{\lambda}{m}\tan\theta_0$.

Here, you can see that a catenary is the image of cosh, which looks like a parabola, but not quite the same.