Optimal Departure Time and Walking Speed for the Cafeteria

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1. Problem

Students may only use the cafeteria floor assigned to their grade. Choose a departure time t and average speed v to minimize the sum of walking and queueing time, food quality loss, and a penalty for walking too fast.

The model makes the following assumptions:

- Arrival times for students in this grade follow a shifted lognormal distribution;
- The cafeteria's maximum service rate is constant;
- Food neither runs out nor is replenished during service;
- Food quality decreases continuously and linearly with the cumulative number of people served;
- The queue is first-come, first-served, and no one leaves before being served.

2. Arrivals and Queueing

Let c be the earliest possible arrival time, and let N be the total number of students in this grade dining on this floor. The cumulative distribution function of arrival times is

$$F(a) = \begin{cases} 0 & a \le c \\ \Phi\left(\frac{\ln(a-c)-m}{\sigma}\right) & a > c \end{cases}.$$

The corresponding density is

$$f(a) = \frac{1}{(a-c)\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(a-c)-m)^2}{2\sigma^2}\right), \quad a > c.$$

Thus, the expected number of arrivals by time a is $NF(a)$. The maximum density is

$$f_{\max} = \frac{\exp\left(-m + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{2\pi}}.$$

Let the constant service rate be μ. If $N f_{\max} \leq \mu$, no queue forms. Otherwise, the queue begins to form at time

$$u = c + \exp\left(m - \sigma^2 - \sigma\sqrt{2\ln\left(\frac{N f_{\max}}{\mu}\right)}\right).$$

The time $e > u$ at which the queue clears is the nontrivial solution of

$$N(F(e) - F(u)) = \mu(e - u)$$

This equation generally has no elementary closed-form solution, but it can be solved using one-dimensional bisection. The queueing time for an arrival at time a is

$$w(a) = \begin{cases} 0 & a \leq u \\ \frac{N}{\mu}(F(a) - F(u)) - (a - u) & u < a \wedge a < e \\ 0 & a \geq e \end{cases}.$$

3. Food Quality

Normalize the initial food quality to 1. Suppose that quality has decreased by a total of η after all N people have been served. The quality after serving n people is then

$$S(n) = 1 - \eta\frac{n}{N}.$$

Under the first-come, first-served approximation, someone arriving at time a has approximately $NF(a)$ people ahead of them, so their food quality loss is

$$1 - S(NF(a)) = \eta F(a).$$

4. Relative Weights

The original penalty function is

$$J(t, v) = \alpha\left(\frac{d}{v} + w\left(t + \frac{d}{v}\right)\right) + \beta\eta F\left(t + \frac{d}{v}\right) + \gamma\left(\frac{(v - v_c)_+}{v_c}\right)^2.$$

Multiplying α, β, γ by the same positive constant does not change the optimum. Thus, when $\alpha > 0$, it suffices to consider two relative weights:

$$r_B = \frac{\beta}{\alpha}, \quad r_V = \frac{\gamma}{\alpha}.$$

After dividing the objective function by α, the function used in the computation is

$$J_{\mathrm{rel}(t,v)} = \frac{d}{v} + w\left(t + \frac{d}{v}\right) + r_B \eta F\left(t + \frac{d}{v}\right) + r_V \left(\frac{(v - v_c)_+}{v_c}\right)^2.$$

This eliminates redundant entries in the three-dimensional table of weights. The configuration file fixes the time weight at 1 and lists the desired values of r_B and r_V separately for tabulation.

5. Solution

Set the arrival time to $a = t + \frac{d}{v}$, and define

$$H(a) = w(a) + r_B \eta F(a),$$

$$C(v) = \frac{d}{v} + r_V \left(\frac{(v - v_c)_+}{v_c}\right)^2.$$

Then $J_{\mathrm{rel}(t,v)} = H(a) + C(v)$. Within the interval $u < a < e$ during which the queue exists,

$$H'(a) = \left(\frac{N}{\mu} + r_B \eta\right) f(a) - 1.$$

Interior stationary points satisfy

$$f(a) = \frac{1}{\frac{N}{\mu} + r_B \eta}.$$

To find the optimal arrival time, it suffices to compare the endpoints of the allowed interval, the queue clearance time e, and any stationary points lying in (u, e).

When $v > v_c$ and $r_V > 0$, the optimal speed satisfies

$$2 r_V (v^*)^2 (v^* - v_c) = d v_c^2.$$

This cubic equation has a unique solution for $v > v_c$; clamp that solution to the allowed speed interval. If $r_V = 0$, choose the maximum allowed speed. The final output is

$$(t^*, v^*) = \left(a^* - \frac{d}{v^*}, v^*\right).$$

6. Parameters

Parameter	Meaning
N	Total number of students in this grade dining on this floor
c	Earliest possible arrival time at the cafeteria
m	Mean of $\ln(a - c)$
σ	Standard deviation of $\ln(a - c)$
μ	Number of people served per minute while a queue exists

Parameter	Meaning
d	Actual travel distance from the departure point to the cafeteria floor assigned to this grade
v_c	Comfortable walking speed
η	Total proportional decrease in quality from the first customer to the last
r_B	Weight of the food quality penalty relative to the time penalty
r_V	Weight of the speed penalty relative to the time penalty

The first eight quantities are obtained through observation or measurement; r_B, r_V are freely chosen personal preferences. The two arrays in `params.example.toml` specify the combinations of relative weights to evaluate.

7. Example Weight Table

The table below is populated automatically from `results.example.csv`, which is generated by the Rust program. Rows correspond to r_B and columns to r_V; each cell lists the optimal departure time followed by the optimal speed (meters per minute).

$r_B \setminus r_V$	0	1	2	4	8
0	12:23:13 160.0	12:23:13 160.0	12:22:30 136.1	12:21:45 117.6	12:21:01 104.0
0.5	12:23:13 160.0	12:23:13 160.0	12:22:30 136.1	12:21:45 117.6	12:21:01 104.0
1	12:23:13 160.0	12:23:13 160.0	12:22:30 136.1	12:21:45 117.6	12:21:01 104.0
2	12:01:56 160.0	12:01:56 160.0	12:01:13 136.1	12:00:28 117.6	11:59:45 104.0
5	12:01:56 160.0	12:01:56 160.0	12:01:13 136.1	12:00:28 117.6	11:59:45 104.0